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Dynamical and symmetry groups of the $SU(2)$ Kepler problem

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Abstract

It is well known that the MIC–Kepler problem, an extension of the three-dimensional Kepler problems, admits the same dynamical and symmetry groups as the Kepler problem. This paper aims to study dynamical and symmetry groups of the $SU(2)$ Kepler problem, where the $SU(2)$ Kepler problem is defined to be the dynamical system reduced from the eight-dimensional conformal Kepler problem through an $SU(2)$ symmetry and turns out to be an extension of the five-dimensional Kepler problem. It is shown that the $SU(2)$ Kepler problem admits a dynamical group $SO^*(8)$ and that the phase space of the $SU(2)$ Kepler problem is symplectomorphic with a co-adjoint orbit of $SO^*(8)$, on which the Kirillov–Kostant–Souriau form is defined. It is further shown that the subgroups, $SU(4)$, $SU^*(4)$, and $Sp(2) \times_S \mathbf{R}^5$, of $SO^*(8)$ provide the symmetry groups, $SU(4)/\mathbf{Z}_2 \cong SO(6)$, $SU^*(4)/\mathbf{Z}_2 \cong SO_0(1,5)$, and $(Sp(2) \times_S \mathbf{R}^5)/\mathbf{Z}_2 \cong SO(5) \times_S \mathbf{R}^5$, of the $SU(2)$ Kepler problem with negative, positive, and zero energies, respectively, where $\times_S$ denotes a semi-direct product. Furthermore, constants of motion for the $SU(2)$ Kepler problem are found together with their Poisson brackets. The symmetry Lie algebra formed by constants of motion is shown to be isomorphic with $so(6) \cong su(4)$, $so(1,5) \cong su^*(4)$, or $so(5) \oplus_S \mathbf{R}^5 \cong sp(2) \oplus_S \mathbf{R}^5$, depending on whether the energy is negative, positive, or zero, where $\oplus_S$ denotes a semi-direct sum. These Lie algebras are subalgebras of $so^*(8) \cong so(2,6)$.

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1. Introduction

The Hopf bundles, $S^3 \to S^2$ and $S^7 \to S^4$, or their respective extensions $\dot{\mathbf{R}}^4 \to \dot{\mathbf{R}}^3$ and $\dot{\mathbf{R}}^8 \to \dot{\mathbf{R}}^5$, have wide application in the study of dynamical systems, where $\dot{\mathbf{R}}^n =$

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$\mathbf{R} - \{0\}$ with $n = 3, 4, 5, 8$. In celestial mechanics, the map $\dot{\mathbf{R}}^4 \to \dot{\mathbf{R}}^3$ is known as the KS (Kustaanheimo–Stiefel) transformation [1], and used for regularizing the Kepler problem [2,3]. This map, viewed as an $SO(2)$ bundle, was also used to set up and analyze the MIC (McIntosh and Cisneros)–Kepler problem [4], which is a generalized Kepler problem defined on the cotangent bundle $T^*\dot{\mathbf{R}}^3$ [5]. This system describes a particle moving in Dirac's monopole field in $\dot{\mathbf{R}}^3$ with the Coulomb potential and a centrifugal potential. The map $\dot{\mathbf{R}}^8 \to \dot{\mathbf{R}}^5$, viewed as an $SU(2)$ bundle, was used to define and analyze the $SU(2)$ Kepler problem [6], which describes a particle moving in Yang's monopole field in $\dot{\mathbf{R}}^5$ [7] along with the Coulomb potential and a centrifugal potential. The definition of this system will be reviewed in this paper.

As is well known, the Kepler problem has a dynamical group $SU(2,2)$, which has long been studied on the Lie group level [3,8–12] as well as on the Lie algebra level [13–19]. According to Iwai [20], the MIC–Kepler problem has the dynamical group $SU(2,2)$ as well. It has subgroups $SU(2) \times SU(2)$, $SL(2,\mathbf{C})$, and $SU(2) \times_S \mathbf{R}^3$, which project to symmetry groups $(SU(2) \times SU(2))/\mathbf{Z}_2 \cong SO(4)$, $SL(2,\mathbf{C})/\mathbf{Z}_2 \cong O_0(3,1)$, and $(SU(2) \times_S \mathbf{R}^3)/\mathbf{Z}_2 \cong SO(3) \times_S \mathbf{R}^3$ of the MIC–Kepler problem with negative, positive, and zero energies, respectively, where $\times_S$ denotes a semi-direct product. To study the symmetry of the MIC–Kepler problem [5,20], extensive use was made of the reduction $T^*\dot{\mathbf{R}}^4 \to T^*\dot{\mathbf{R}}^3$. This reduction method was also used for the study of Kepler-type symmetries of other dynamical systems by Cordani et al. [21]. They provided a method for constructing a dynamical system which has a manifest $su(2,2)$ symmetry. Toshihiro Iwai also used the reduction method to study the Kepler-type symmetry from the viewpoint of closedness of bounded orbits [22–24].

The MIC–Kepler problem is already generalized to the $SU(2)$ Kepler problem [6], the symmetry of which can be studied in parallel with the MIC–Kepler problem. The present paper has an aim to show that the $SU(2)$ Kepler problem has a dynamical group $Sp(2\mathrm{i}, 2\mathrm{i})$, which is defined to be

$$Sp(2\mathrm{i}, 2\mathrm{i}) = \{g \in M(4, \mathbf{H});\ \ gGg^* = G,\ \ G = \mathrm{diag}(\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i})\},$$

and shown to be isomorphic with $SO^*(8)$, if restricted to the identity component, where $SO^*(8)$ is defined, in general, to be

$$SO^*(2n) = \{g \in M(2n, \mathbf{C});\ \ J_n g = \bar{g} J_n,\ \ {}^t g g = I_{2n},\ \ \det\ g = 1\} \tag{i}$$

along with I_{2n} the $2n \times 2n$ identity matrix and

$$J_n = \mathrm{diag}(J_1, \ldots, J_1), \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The notation $Sp(2\mathrm{i}, 2\mathrm{i})$ is provisional in this paper. The dynamical group $Sp(2\mathrm{i}, 2\mathrm{i})$ proves to have subgroups $SU(4)$, $SU^*(4)$, and $Sp(2) \times_S \mathbf{R}^5$, which will project to the symmetry groups of the $SU(2)$ Kepler problem with energy negative, positive, and zero, respectively, where

$$SU^*(2n) = \{g \in M(2n, \mathbf{C});\ \ J_n g = \bar{g} J_n,\ \ \det\ g = 1\}. \tag{ii}$$

See [25] for the definition of $SO^*(2n)$ and $SU^*(2n)$, for example. Like the MIC–Kepler problem, the symmetry groups of the $SU(2)$ Kepler problem will be obtained by factoring the above subgroups by $\mathbf{Z}_2$.

The organization of this paper is as follows: Section 2 contains a review of the $SU(2)$ bundle, $\mathbf{R}^8 \to \dot{\mathbf{R}}^5 := \mathbf{R}^5 - \{0\}$, and of the $SU(2)$ Kepler problem. The $SU(2)$ Kepler problem is defined to be the reduced Hamiltonian system of the conformal Kepler problem on $(T^*(\dot{\mathbf{R}}^8), \mathrm{d}\theta)$ with $\mathrm{d}\theta$ the standard symplectic form.

Section 3 shows that $Sp(2\mathrm{i}, 2\mathrm{i})$ acts symplectically on $(T^*(\dot{\mathbf{R}}^8), \mathrm{d}\theta)$, and further studies the momentum map associated with $Sp(2\mathrm{i}, 2\mathrm{i})$. It then turns out that the $Sp(2\mathrm{i}, 2\mathrm{i})$ should be called a dynamical group of the conformal Kepler problem. In fact, some of one-parameter subgroups of $Sp(2\mathrm{i}, 2\mathrm{i})$ are associated with the Hamiltonian flow of the conformal Kepler problem with energy values fixed. The $Sp(2\mathrm{i}, 2\mathrm{i})$ is further shown to provide a dynamical group of the $SU(2)$ Kepler problem as well. Moreover, the phase space of the $SU(2)$ Kepler problem is shown to be symplectomorphic with a co-adjoint orbit of $Sp(2\mathrm{i}, 2\mathrm{i})$, on which the Kirillov–Kostant–Souriau form is defined [26,27].

Section 4 shows that the dynamical group $Sp(2\mathrm{i}, 2\mathrm{i})$ contains subgroups $SU(4)$, $SU^*(4)$, and $Sp(2) \times_S \mathbf{R}^5$, which project, respectively, to the symmetry groups $SU(4)/\mathbf{Z}_2 \cong SO(6)$, $SU^*(4)/\mathbf{Z}_2 \cong O_0(5,1)$, and $(Sp(2) \times_S \mathbf{R}^5)/\mathbf{Z}_2 \cong SO(5) \times_S \mathbf{R}^5$ for the $SU(2)$ Kepler problem of negative, positive, and zero energies.

Section 5 deals with the constants of motion. By using the Lie algebras of the subgroups of $Sp(2\mathrm{i}, 2\mathrm{i})$ found in Section 4, constants of motion for the conformal Kepler problem are first obtained together with their Poisson brackets, and then they project to constants of motion for the $SU(2)$ Kepler problem. It will be shown that the symmetry Lie algebra consisting of constants of motion is isomorphic with $so(6) \cong su(4)$, $so(1,5) \cong su^*(4)$, or $e(5) \cong so(5) \oplus_S \mathbf{R}^5$, according to whether the energy is negative, positive, or zero. These Lie algebras are subalgebras of the dynamical Lie algebra $so^*(8) \cong so(2,6)$. Section 6 contains concluding remarks.

2. A review of the $SU(2)$ Kepler problem

This section gives a review of the definition of the $SU(2)$ Kepler problem as well as of the reduction of the phase space $T^*(\dot{\mathbf{R}}^8)$ after Iwai [6], but the notations used here are a bit different from those in [6]. Let (x_m, y_n), $m, n = 0, \ldots, 7$, be the Cartesian coordinates of the phase space $T^*(\dot{\mathbf{R}}^8) = \dot{\mathbf{R}}^8 \times \mathbf{R}^8$. To give the concise description of the reduction procedure, it is convenient for us to introduce quaternions

$$q_1 = x_0 + x_1 i + x_2 j + x_3 k, \quad q_2 = x_4 + x_5 i + x_6 j + x_7 k,$$
$$p_1 = y_0 + y_1 i + y_2 j + y_3 k, \quad p_2 = y_4 + y_5 i + y_6 j + y_7 k, \tag{1}$$

through which we can identify $\mathbf{R}^8 \times \mathbf{R}^8$ with $\mathbf{H}^2 \times \mathbf{H}^2$.

Before carrying out the reduction procedure, we have to make a brief review of the quaternion algebra $\mathbf{H}$. Let

$$MJ(2,\mathbf{C}) = \left\{ \begin{pmatrix} x_0 + x_1\mathrm{i} & x_2 + x_3\mathrm{i} \\ -x_2 + x_3\mathrm{i} & x_0 - x_1\mathrm{i} \end{pmatrix};\ x_0 + x_1\mathrm{i},\ x_2 + x_3\mathrm{i} \in \mathbf{C} \right\}, \tag{2}$$

which is characterized as

$$MJ(2,\mathbf{C}) = \{A \in M(2,\mathbf{C});\ E_2 A = \bar{A} E_2\}, \tag{3}$$

where $\bar{A}$ is the complex conjugate of A, and E_2 will soon be defined in (5). Then the following map gives the standard $\mathbf{R}$-vector space isomorphism of $\mathbf{H}$ with $MJ(2,\mathbf{C})$,

$$\rho(x_0 + x_1 i + x_2 j + x_3 k) = x_0 E_0 + x_1 E_1 + x_2 E_2 + x_3 E_3,\quad x_\nu \in \mathbf{R},\quad \nu = 0,\ldots,3, \tag{4}$$

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad E_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},\quad E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad E_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{5}$$

The map ρ is an algebra isomorphism of $\mathbf{H}$ with $MJ(2,\mathbf{C})$ as well. In particular, one has

$$\rho(q_1 q_2) = \rho(q_1)\rho(q_2),\quad q_1, q_2 \in \mathbf{H}. \tag{6}$$

Note further that the quaternion conjugate is mapped to the matrix Hermitian conjugate

$$\rho(q^*) = \rho(q)^*,\quad q \in \mathbf{H}. \tag{7}$$

Then the restriction of ρ to the subset of $h \in \mathbf{H}$ with $h^*h = 1$ yields the group isomorphism

$$Sp(1) \cong SU(2) \cong SU^*(2), \tag{8}$$

where $SU^*(2)$ is the $MJ(2,\mathbf{C})$ restricted by $\det g = 1$ with $g \in MJ(2,\mathbf{C})$.

Now we consider $\mathbf{H}^2$ as a real vector space $\mathbf{R}^8$. Then the standard inner product on $\mathbf{R}^8$ brings about the inner product on $\mathbf{H}^2$; for $q = (q_1, q_2)$, $p = (p_1, p_2) \in \mathbf{H}^2$, their inner product $q \cdot p$ is put in the form

$$q \cdot p = \frac{1}{2}\sum_{\lambda=1}^{2}(q_\lambda p_\lambda^* + p_\lambda q_\lambda^*), \tag{9}$$

which we can express as

$$q \cdot p = \tfrac{1}{2}(qp^* + pq^*), \tag{10}$$

where q and p^* denote row and column vectors, respectively, so that qp^* and pq^* are scalars. Alternatively, the inner product is expressed as

$$q \cdot p = \frac{1}{2}\sum_{\lambda=1}^{2}(q_\lambda^* p_\lambda + p_\lambda^* q_\lambda) = \frac{1}{2}\mathrm{tr}(q^*p + p^*q), \tag{11}$$

where q^*p and q^*p denote 2×2 matrices.

The group $Sp(1) \cong SU(2)$ acts on $\mathbf{H}^2$ to the left in a natural manner

$$(q_1, q_2) \mapsto (hq_1, hq_2),\quad h \in Sp(1). \tag{12}$$

From (11), it follows that the inner product in $\mathbf{H}^2$ is invariant under the left $Sp(1)$ action

$$hq \cdot hp = q \cdot p, \quad p, q \in \mathbf{H}^2, \ h \in Sp(1). \tag{13}$$

Since $Sp(1)$ is compact and since the $Sp(1)$ left action on $\dot{\mathbf{H}}^2 := \mathbf{H}^2 - \{0\}$ is free, the space $\dot{\mathbf{H}}^2 \cong \dot{\mathbf{R}}^8$ is made into a principal $Sp(1)$ bundle, of which the phase space is isomorphic with $\dot{\mathbf{R}}^5$

$$Sp(1) \to \dot{\mathbf{H}}^2 \to \dot{\mathbf{R}}^5. \tag{14}$$

Here, the projection $\dot{\mathbf{H}}^2 \to \dot{\mathbf{R}}^5$ is realized as

$$\pi : \ (q_1, q_2) \mapsto (2q_1^* q_2, \ q_1^* q_1 - q_2^* q_2) \in \mathbf{H} \times \mathbf{R}. \tag{15}$$

We here set $2q_1^* q_2 = u_0 + u_1 i + u_2 j + u_3 k$ and $q_1^* q_1 - q_2^* q_2 = u_4$. Then the projection π is expressed as

$$\begin{aligned} &u_0 = 2(x_0x_4 + x_1x_5 + x_2x_6 + x_3x_7), \quad u_1 = 2(x_0x_5 - x_1x_4 - x_2x_7 + x_3x_6),\\ &u_2 = 2(x_0x_6 + x_1x_7 - x_2x_4 - x_3x_5), \quad u_3 = 2(x_0x_7 - x_1x_6 + x_2x_5 - x_3x_4),\\ &u_4 = x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2. \end{aligned} \tag{16}$$

Further, a simple calculation gives

$$\left(\sum_{m=0}^{7} x_m^2\right)^2 = \left(|q_1|^2 + |q_2|^2\right)^2 = \sum_{\ell=0}^{4} u_\ell^2 =: r^2. \tag{17}$$

The infinitesimal generators of the $Sp(1)$ left action are expressed as ζq with $\zeta \in sp(1)$ and $q \in \dot{\mathbf{H}}^2$, where $sp(1)$ is the Lie algebra of $Sp(1)$, the set of all the pure imaginary quaternions. The vector fields ζq's, called vertical (or fundamental) vector fields, are tangent to fibers of the principal $Sp(1)$ bundle. Those vector fields that are orthogonal to vertical vector fields with respect to the standard metric on $\dot{\mathbf{R}}^8 \cong \dot{\mathbf{H}}^2$ are called horizontal vector fields. There are mutually orthogonal vector fields, three of which are vertical ones, V_1, V_2, V_3, and the other five of which are horizontal ones, $H_0, H_1, \ldots, H_4$; they are expressed, in components, as

$$\begin{aligned} V_1 &= (-x_1, x_0, -x_3, x_2, -x_5, x_4, -x_7, x_6),\\ V_2 &= (-x_2, x_3, x_0, -x_1, -x_6, x_7, x_4, -x_5),\\ V_3 &= (-x_3, -x_2, x_1, x_0, -x_7, -x_6, x_5, x_4),\\ H_0 &= (x_4, x_5, x_6, x_7, x_0, x_1, x_2, x_3),\\ H_1 &= (x_5, -x_4, -x_7, x_6, -x_1, x_0, x_3, -x_2),\\ H_2 &= (x_6, x_7, -x_4, -x_5, -x_2, -x_3, x_0, x_1),\\ H_3 &= (x_7, -x_6, x_5, -x_4, -x_3, x_2, -x_1, x_0),\\ H_4 &= (x_0, x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7). \end{aligned} \tag{18}$$

At each point $q \in \dot{\mathbf{H}}^2$, the horizontal subspace spanned by H_ℓ, $\ell = 0, \dots, 4$, defines a connection on the $Sp(1)$ bundle. In a dual manner, the connection form ω is defined to be

$$\omega = \frac{1}{2r}(\mathrm{d}q_1\, q_1^* - q_1\, \mathrm{d}q_1^* + \mathrm{d}q_2\, q_2^* - q_2\, \mathrm{d}q_2^*), \tag{19}$$

where r is the radial variable given in (17). Clearly, the form ω satisfies

$$\omega_{hq} = \mathrm{Ad}_h \omega_q, \quad q \in \dot{\mathbf{H}}^2, \ h \in Sp(1), \tag{20}$$

$$\omega_q(\zeta q) = \zeta, \quad q \in \dot{\mathbf{H}}^2, \ \zeta \in sp(1). \tag{21}$$

Note also that the horizontal subspace at $q \in \dot{\mathbf{H}}^2$ is given by $\ker \omega_q$, where ω_q is looked upon as a linear map $T_q(\dot{\mathbf{H}}^2) \to sp(1)$. If we set $\omega = \omega_1 i + \omega_2 j + \omega_3 k$, the components ω_α, $\alpha = 1, 2, 3$, have the expression

$$\omega_\alpha = \frac{1}{r} V_\alpha \cdot \mathrm{d}q, \quad \alpha = 1, 2, 3, \tag{22}$$

where V_α are the vertical vector fields defined in (18) and $\mathrm{d}q$ is viewed as an $\mathbf{R}^8$-valued one-form. For the sake of comparison with ω_α, we obtain, from (16),

$$\mathrm{d}u_\ell = 2H_\ell \cdot \mathrm{d}q, \quad \ell = 0, 1, \dots, 4, \tag{23}$$

where H_ℓ are the horizontal vector fields defined also in (18).

The left action of $Sp(1)$ on $\dot{\mathbf{H}}^2$ lifts naturally to a left action T_h on the phase space $T^*(\dot{\mathbf{R}}^8) \cong \dot{\mathbf{H}}^2 \times \mathbf{H}^2$; for $q = (q_1, q_2) \in \dot{\mathbf{H}}^2$, $p = (p_1, p_2) \in \mathbf{H}^2$, one has

$$T_h : (q, p) \mapsto (hq, hp). \tag{24}$$

The standard symplectic form on $T^*(\dot{\mathbf{R}}^8)$ is defined to be and expressed, in terms of quaternions, as

$$\mathrm{d}\theta = \sum_{m=0}^{7} \mathrm{d}y_m \wedge \mathrm{d}x_m = \frac{1}{2}\sum_{\lambda=1}^{2}(\mathrm{d}p_\lambda \wedge \mathrm{d}q_\lambda^* - \mathrm{d}q_\lambda \wedge \mathrm{d}p_\lambda^*), \tag{25}$$

with

$$\theta = \sum_{m=0}^{7} y_m\, \mathrm{d}x_m = \frac{1}{2}\sum_{\lambda=1}^{2}(p_\lambda\, \mathrm{d}q_\lambda^* + \mathrm{d}q_\lambda\, p_\lambda^*). \tag{26}$$

We can put the $\mathrm{d}\theta$ in the form

$$\mathrm{d}\theta = \tfrac{1}{2}(\mathrm{d}p \wedge \mathrm{d}q^* - \mathrm{d}q \wedge \mathrm{d}p^*), \tag{27}$$

where $\mathrm{d}p$ and $\mathrm{d}q^*$ are row and column vectors of one-forms, respectively, so that $\mathrm{d}p \wedge \mathrm{d}q^*$ and $\mathrm{d}q \wedge \mathrm{d}p^*$ are scalar-valued two-forms. An alternative expression of $\mathrm{d}\theta$ is given by

$$\mathrm{d}\theta = \frac{1}{2}\sum_{\lambda=1}^{2}(\mathrm{d}p_\lambda^* \wedge \mathrm{d}q_\lambda - \mathrm{d}q_\lambda^* \wedge \mathrm{d}p_\lambda) = \frac{1}{2}\mathrm{tr}(\mathrm{d}p^* \wedge \mathrm{d}q - \mathrm{d}q^* \wedge \mathrm{d}p), \tag{28}$$

where $\mathrm{d}p^* \wedge \mathrm{d}q$ and $\mathrm{d}q^* \wedge \mathrm{d}p$ denote matrix-valued two-forms. On introducing $z = (q, p) \in \dot{\mathbf{H}}^2 \times \mathbf{H}^2$, the $\mathrm{d}\theta$ given in (27) can be expressed as

$$\mathrm{d}\theta = -\tfrac{1}{2}\mathrm{d}z \wedge J\mathrm{d}z^*, \quad z = (q, p), \quad J = \begin{pmatrix} 0 & E_0 \\ -E_0 & 0 \end{pmatrix}, \tag{29}$$

where $\mathrm{d}z \wedge J\mathrm{d}z^*$ is a scalar-valued two-form. Alternatively, one has, from (28)

$$\mathrm{d}\theta = \tfrac{1}{2}\mathrm{tr}(J\mathrm{d}z^* \wedge \mathrm{d}z), \tag{30}$$

where $J\mathrm{d}z^* \wedge \mathrm{d}z$ is a matrix-valued two-form. From this, it follows that the symplectic form $\mathrm{d}\theta$ is invariant under the $Sp(1)$ left action T_h.

The Lie algebra $su(2)$ of $SU(2)$ is endowed with the inner product

$$\langle \xi | \eta \rangle = \tfrac{1}{2}\,\mathrm{tr}\,\xi\eta^*, \quad \xi, \eta \in su(2), \tag{31}$$

through which we identify $su(2)^*$ with $su(2)$. Since $su(2)$ is isomorphic with $sp(1)$, there is defined the corresponding inner product in $sp(1)$, which is expressed as and equal to

$$\mathrm{Re}(\rho^{-1}(\xi)\rho^{-1}(\eta^*)) = \langle \xi | \eta \rangle, \quad \xi, \eta \in su(2), \tag{32}$$

where Re denotes the real part, and $\rho: sp(1) \to su(2)$ is the algebra isomorphism $\rho: \mathbf{H} \to MJ(2, \mathbf{C})$ restricted to $sp(1)$.

Now we are in a position to carry out the reduction procedure. We begin by finding the momentum map associated with the $Sp(1)$ left action. To this end, we introduce a one-form

$$\Theta = \tfrac{1}{4}(p\,\mathrm{d}q^* + \mathrm{d}q\,p^* - q\,\mathrm{d}p^* - \mathrm{d}p\,q^*), \tag{33}$$

which satisfies $\mathrm{d}\theta = \mathrm{d}\Theta$ on account of (27). The infinitesimal generator of the $Sp(1)$ left action on $P := \dot{\mathbf{H}}^2 \times \mathbf{H}^2$ takes the form $\zeta_P = (\zeta q, \zeta p)$ with $\zeta \in sp(1)$ and $(q, p) \in \dot{\mathbf{H}}^2 \times \mathbf{H}^2$. Then the function $\Theta(\zeta_P)$ provides the generating function of ζ_P and is expressed as

$$\Theta(\zeta_P) = \tfrac{1}{4}((pq^* - qp^*)\zeta^* + \zeta(qp^* - pq^*)), \tag{34}$$

from which we observe that the momentum map, $\Psi : \dot{\mathbf{H}}^2 \times \mathbf{H}^2 \to sp(1)$, associated with the $Sp(1)$ left action takes the form

$$\Psi(q, p) = \tfrac{1}{2}(pq^* - qp^*). \tag{35}$$

It is an immediate consequence of (35) that the momentum map Ψ is Ad-equivariant

$$\Psi \circ T_h = \mathrm{Ad}_h \Psi, \quad h \in Sp(1). \tag{36}$$

If we set $\Psi = \Psi_1 i + \Psi_2 j + \Psi_3 k$, the components Ψ_α are put in the form

$$\Psi_\alpha = p \cdot V_\alpha, \quad \alpha = 1, 2, 3, \tag{37}$$

where p is viewed as a vector in $\mathbf{R}^8$ and V_α are the vertical vector fields defined in (18).

According to the Weinstein–Marsden reduction procedure [28], the reduction of the phase space $(T^*(\dot{\mathbf{H}}^2), \mathrm{d}\theta)$ runs as follows: A momentum manifold $\Psi^{-1}(\mu)$ for a fixed $\mu \in sp(1)$ with $\mu \neq 0$ projects to the reduced phase space defined as

$$P_\mu := \Psi^{-1}(\mu)/G_\mu, \quad G_\mu = \{h \in Sp(1);\ \ \mathrm{Ad}_h\mu = \mu\}, \tag{38}$$

where G_μ, called the isotropy subgroup, is isomorphic to $U(1)$ when $\mu \neq 0$. Let ι_μ and π_μ be the inclusion map $\iota_\mu : \ \Psi^{-1}(\mu) \to T^*(\dot{\mathbf{H}}^2)$ and the natural projection $\pi_\mu : \Psi^{-1}(\mu) \to \Psi^{-1}(\mu)/G_\mu$, respectively. The reduced symplectic form σ_μ on P_μ is then determined through

$$\pi_\mu^*\sigma_\mu = \iota_\mu^*\,\mathrm{d}\theta. \tag{39}$$

If $\mu \neq 0$, the reduced phase space is not isomorphic with $T^*(\dot{\mathbf{R}}^5)$, but isomorphic with a fiber bundle with base space $T^*(\dot{\mathbf{R}}^5)$ and fiber $Sp(1)/U(1) \cong S^2$. For details, see Iwai [6].

In view of (37), we introduce variables v_ℓ by

$$v_\ell = \frac{1}{2r}p \cdot H_\ell, \quad \ell = 0, 1, \dots, 4, \tag{40}$$

where H_ℓ are the horizontal vector fields defined in (18). Then the standard one-form θ turns out to be expressed as

$$\theta = p \cdot \mathrm{d}q = \sum_{\ell=0}^{4} v_\ell\,\mathrm{d}u_\ell + \sum_{\alpha=1}^{3}\Psi_\alpha\omega_\alpha, \tag{41}$$

where use has been made of (22),(23),(37) and (40). Accordingly, one has, on $\Psi^{-1}(\mu)$,

$$\iota_\mu^*\theta = \sum_{\ell=0}^{4} v_\ell\,\mathrm{d}u_\ell + \sum_{\alpha=1}^{3}\mu_\alpha\iota_\mu^*\omega_\alpha, \tag{42}$$

where μ_α are the components of $\mu \in sp(1)$, and ω_α are viewed as pulled-back on $T^*(\dot{\mathbf{R}}^8)$ through the projection $T^*(\dot{\mathbf{R}}^8) \to \dot{\mathbf{R}}^8$. Thus the reduced symplectic form σ_μ is determined through

$$\pi_\mu^*\sigma_\mu = \sum_{\ell=0}^{4}\mathrm{d}v_\ell \wedge \mathrm{d}u_\ell + \sum_{\alpha=1}^{3}\mu_\alpha\,\mathrm{d}\iota_\mu^*\omega_\alpha. \tag{43}$$

By definition, (u_ℓ, v_ℓ) can be viewed as coordinates of $T^*(\dot{\mathbf{R}}^5)$, so that the first term in the right-hand side of (43) can project to the standard symplectic form on $T^*(\dot{\mathbf{R}}^5)$.

Now we are in a position to give a concise definition of the $SU(2)$ Kepler problem. The $SU(2)$ Kepler problem is the Hamiltonian system which is reduced from the conformal Kepler problem on $T^*(\dot{\mathbf{R}}^8)$ by using the $SU(2) \cong Sp(1)$ symmetry. The Hamiltonian H_C of the conformal Kepler problem is given by

$$H_\mathrm{C} = \frac{1}{2}\frac{1}{4r}p \cdot p - \frac{\kappa}{r}, \tag{44}$$

where $(q, p) \in \dot{\mathbf{H}}^2 \times \mathbf{H}^2 = T^*(\dot{\mathbf{H}}^2)$, $r = q \cdot q = \sum_{m=0}^{7} x_m^2$, and κ is a positive constant. Thus the $SU(2)$ Kepler problem is defined as a triple $(P_\mu, \sigma_\mu, H_\mu)$, where H_μ is the Hamiltonian determined through $H_C \circ \iota_\mu = H_\mu \circ \pi_\mu$. From (44) along with (37) and (40), H_μ turns out to be expressed as

$$H_\mu = \frac{1}{2}\sum_{\ell=0}^{4} v_\ell^2 + \frac{|\mu|^2}{8r^2} - \frac{\kappa}{r}, \quad |\mu|^2 = \sum_{\alpha=1}^{3} \mu_\alpha^2. \tag{45}$$

We notice here that the $SU(2)$ Kepler problem has been already quantized on the vector bundle associated with the $Sp(1)$ bundle (14) [29].

3. A dynamical group

3.1. A group of linear symplectic transformations

From the expression (29) of the symplectic form $\mathrm{d}\theta$, we observe that the right action of a quaternionic 4×4 matrix g preserves the symplectic form, if and only if it satisfies $gJg^* = J$. Thus we obtain the group of linear symplectic transformations

$$GJ(4, \mathbf{H}) = \{g \in M(4, \mathbf{H});\ \ gJg^* = J\}, \tag{46}$$

where the notation $GJ(4, \mathbf{H})$ is provisional in this paper. Since left and right actions commute and since both of the $Sp(1)$ left action and the $GJ(4, \mathbf{H})$ right action leave the symplectic form $\mathrm{d}\theta$ invariant, the product group $Sp(1) \times GJ(4, \mathbf{H})$ acts symplectically on the phase space.

In order to consider which classical group the $GJ(4, \mathbf{H})$ is isomorphic to, we are going to express $GJ(4, \mathbf{H})$ in the complex matrix form. The algebra isomorphism ρ, given in (4), of $\mathbf{H}$ with $MJ(2, \mathbf{C})$ lifts to an algebra isomorphism $\rho : M(2, \mathbf{H}) \to MJ(4, \mathbf{C})$ in a natural manner

$$\rho\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} = \begin{pmatrix} \rho(q_1) & \rho(q_2) \\ \rho(q_3) & \rho(q_4) \end{pmatrix}, \quad q_1, \dots, q_4 \in \mathbf{H}, \tag{47}$$

where $\rho(q_1) \in MJ(2, \mathbf{C})$, etc. and $MJ(4, \mathbf{C})$ is the set of matrices defined to be

$$MJ(4, \mathbf{C}) = \{h \in M(4, \mathbf{C});\ \ hJ_d = J_d\bar{h}\}, \quad J_d = \begin{pmatrix} E_2 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{48}$$

In a similar manner, ρ lifts to an algebra isomorphism $M(4, \mathbf{H}) \to MJ(8, \mathbf{C})$, where $MJ(8, \mathbf{C})$ is defined like in (48). Since J and J_d are similar, i.e., one has $J = T^{-1}J_dT$ for an invertible matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{49}$$

the condition in (46) turns out to be equivalently expressed, in terms of complex matrices, as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} J_d & 0 \\ 0 & J_d \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} J_d & 0 \\ 0 & J_d \end{pmatrix}, \tag{50}$$

where $A, B, C, D \in MJ(4, \mathbf{C})$. From the condition (48), the matrices A, B, C, D are subject to $A^* = -J_d^{\mathrm{t}} A J_d$, etc. so that the condition (50) is brought into the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix} = I_8, \tag{51}$$

where I_8 denotes the 8×8 identity matrix. The conditions (50) and (51) along with

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$$

are put together to define the group $SO^*(8)$ (see (i) in Section 1 for the definition of $SO^*(2n)$). Thus we have obtained the isomorphism

$$GJ(4, \mathbf{H})_0 \cong SO^*(8), \tag{52}$$

where the subscript 0 denotes the identity component. It is well known that $\dim SO^*(2n) = n(2n-1)$ [25], so that $\dim SO^*(8) = 28$. It is also known that the Lie algebra $so^*(8)$ of $SO^*(8)$ is isomorphic to $so(2, 6)$ [25]

$$so^*(8) \cong so(2, 6). \tag{53}$$

See also Yokota and Miyashita [30] for the proof of the isomorphism (53) in the explicit form.

3.2. The associated momentum map

We proceed to the momentum map associated with $GJ(4, \mathbf{H})$. To this end, let us introduce a new variable w by

$$w = zS, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} E_0 & E_0 \\ \mathrm{i}E_0 & -\mathrm{i}E_0 \end{pmatrix}. \tag{54}$$

Then the $\mathrm{d}\theta$ given in (29) takes the form

$$\mathrm{d}\theta = -\tfrac{1}{2}\, \mathrm{d}w \wedge G\, \mathrm{d}w^*, \tag{55}$$

where

$$G = \begin{pmatrix} \mathrm{i}E_0 & 0 \\ 0 & -\mathrm{i}E_0 \end{pmatrix}. \tag{56}$$

Alternatively, one obtains, from (28) and (54)

$$d\theta = \frac{1}{4}\,\mathrm{tr}(G\,dw^* \wedge dw + dw^* \wedge dw G). \tag{57}$$

In view of this, we introduce a one-form

$$\Theta' = \frac{1}{4}\mathrm{tr}(Gw^*\,dw - dw^* wG), \tag{58}$$

which satisfies $d\Theta' = d\theta$. Further, the momentum map Ψ given in (35) is then expressed as

$$\Psi(z) = -\frac{1}{2}zJz^* \quad \text{or} \quad \Phi(w) = -\frac{1}{2}wGw^*. \tag{59}$$

It is convenient for us to work with the phase space $(T^*(\dot{\mathbf{H}}^2), d\Theta')$ in the following.

From (46) and (54), it follows that the group $GJ(4,\mathbf{H})$ is brought into the form

$$S^{-1}GJ(4,\mathbf{H})S = Sp(2\mathrm{i}, 2\mathrm{i}) := \{g \in M(4,\mathbf{H});\ \ gGg^* = G\}. \tag{60}$$

The notation $Sp(2\mathrm{i}, 2\mathrm{i})$ is provisional, but we choose to use it for convenience in comparison with $Sp(n,m)$. From (52) and (60), we have $Sp(2\mathrm{i}, 2\mathrm{i})_0 \cong SO^*(8)$, where the subscript 0 indicates the identity components of the group. In what follows, we study the momentum map associated with $Sp(2\mathrm{i}, 2\mathrm{i})$. We denote the Lie algebra of $Sp(2\mathrm{i}, 2\mathrm{i})$ by $sp(2\mathrm{i}, 2\mathrm{i})$, which is isomorphic with $so^*(8) \cong so(2,6)$ (see (53)). A matrix $\xi \in M(4,\mathbf{H})$ is in $sp(2\mathrm{i}, 2\mathrm{i})$, if and only if it satisfies $\xi G + G\xi^* = 0$. A basis of $sp(2\mathrm{i}, 2\mathrm{i})$ and the commutation relations among them are given in Appendices A and B, respectively. The Lie algebra $sp(2\mathrm{i}, 2\mathrm{i})$ is endowed with the inner product

$$(\xi|\eta) = \frac{1}{2}\mathrm{tr}(\xi^*\eta + \eta^*\xi), \quad \xi,\ \eta \in sp(2\mathrm{i}, 2\mathrm{i}), \tag{61}$$

through which the dual space $sp(2\mathrm{i}, 2\mathrm{i})^*$ is identified with $sp(2\mathrm{i}, 2\mathrm{i})$.

We are now in a position to determine the momentum map associated with the $Sp(2\mathrm{i}, 2\mathrm{i})$ right action. With $\xi \in sp(2\mathrm{i}, 2\mathrm{i})$ is associated the infinitesimal generator $\xi_P = w\xi$. Then from (58), we obtain the generating function of ξ_P

$$\Theta'(\xi_P) = \frac{1}{4}\mathrm{tr}(Gw^*w\xi - \xi^*w^*wG). \tag{62}$$

Hence, the momentum map, $K : T^*(\dot{\mathbf{H}}^2) \to sp(2\mathrm{i}, 2\mathrm{i})$, associated with the $Sp(2\mathrm{i}, 2\mathrm{i})$ right action proves to be expressed as

$$K(w) = -\frac{1}{2}w^*wG, \tag{63}$$

where use has been made of the fact that $G^* = -G$. It follows from this that K is subject to the transformation

$$\begin{aligned} K(wg) &= \mathrm{Ad}_{g^*}K(w), \quad && g \in Sp(2\mathrm{i}, 2\mathrm{i}),\\ K(hw) &= K(w), && h \in Sp(1). \end{aligned} \tag{64}$$

In comparison with this, we obtain, from (59),

$$\begin{aligned} \Phi(wg) &= \Phi(g), \quad && g \in Sp(2\mathrm{i}, 2\mathrm{i}),\\ \Phi(hw) &= \mathrm{Ad}_h\Phi(w), && h \in Sp(1). \end{aligned} \tag{65}$$

3.3. Co-adjoint orbits

This subsection shows that the reduced phase space P_μ is realized as a co-adjoint orbit of $Sp(2\mathrm{i}, 2\mathrm{i})$. We begin with the following Lemma, the proof of which will be given in Appendix D.

Lemma 1. *$Sp(2\mathrm{i}, 2\mathrm{i})$ acts transitively on $\Phi^{-1}(\mu) \cong \Psi^{-1}(\mu)$ with $\mu \neq 0$.*

Take a point $p_0 \in \Phi^{-1}(\mu)$ with $\mu \neq 0$. The above lemma then implies that

$$\Phi^{-1}(\mu) = \{p_0 g;\ \ g \in Sp(2\mathrm{i}, 2\mathrm{i})\}, \tag{66}$$

which is mapped, through the momentum map K, to

$$K(\Phi^{-1}(\mu)) = \{\mathrm{Ad}_{g^*} K(p_0);\ \ g \in Sp(2\mathrm{i}, 2\mathrm{i})\}. \tag{67}$$

Thus $K(\Phi^{-1}(\mu))$ becomes a co-adjoint orbit in $sp(2\mathrm{i}, 2\mathrm{i})$, which we denote by $\mathcal{O}_{K(p_0)}$. Now we can prove the following theorem.

Theorem 2. *For $\mu \neq 0$, the reduced phase space (P_μ, σ_μ) is symplectomorphic with a co-adjoint orbit $(\mathcal{O}_{K(p_0)}, \tau)$ of $Sp(2\mathrm{i}, 2\mathrm{i})$, where τ is the Kirillov–Kostant–Souriau (KKS) form.*

Proof. To start with, we show that the orbit $\mathcal{O}_{K(p_0)}$ is diffeomorphic to the reduced phase space P_μ. Let $\iota_\mu : \Phi^{-1}(\mu) \to T^*(\dot{\mathbf{H}}^2)$ be the inclusion map, and consider the composite map given by $K \circ \iota_\mu : \Phi^{-1}(\mu) \to \mathcal{O}_{K(p_0)}$. Suppose that $K(w) = K(w')$ for $w, w' \in \Phi^{-1}(\mu)$. Then one has $w^* w G = w'^* w' G$ together with $\Phi(w) = \Phi(w') = \mu$, from which it follows that w and w' are related by $w' = hw$ for some $h \in Sp(1)$ with $\mathrm{Ad}_h \mu = \mu$, where use has been made of (65). Hence one verifies that $[w] = [w']$ in $\Phi^{-1}(\mu)/G_\mu$, where [ ] denotes the equivalence class. Thus $K \circ \iota_\mu$ induces the diffeomorphism

$$\tilde{K}_\mu :\ \ P_\mu = \Phi^{-1}(\mu)/G_\mu \to \mathcal{O}_{K(p_0)}, \tag{68}$$

through $\tilde{K}_\mu([w]) = K \circ \iota_\mu(w)$. In other words, one has

$$\tilde{K}_\mu \circ \pi_\mu = K \circ \iota_\mu. \tag{69}$$

To accomplish the proof, we have only to show that $\tilde{K}_\mu$ is a symplectic map. As is well known, any co-adjoint orbit is endowed with a symplectic form, called the Kirillov–Kostant–Souriau (KKS) form [26,27]. The KKS form τ on $\mathcal{O}_{K(p_0)}$ is defined, at $\nu \in \mathcal{O}_{K(p_0)}$, to be

$$\tau(\xi_Q, \eta_Q)(\nu) = (\nu |\, [\xi, \eta]), \tag{70}$$

where ξ_Q and η_Q are the infinitesimal generators of the co-adjoint action of $\exp(t\xi)$ and of $\exp(t\eta)$ on $Q := sp(2\mathrm{i}, 2\mathrm{i})$, respectively, and $(|)$ denotes the inner product defined in (61). From (64), ξ_Q is related to ξ_P by

$$K_* \xi_P(w) = \xi_Q(\nu) \ \text{ with } \ \nu = K(w), \quad w \in P = \dot{\mathbf{H}}^2 \times \mathbf{H}^2, \tag{71}$$

where K_* denotes the differential of K. For ξ_P and η_P, and for $w \in \Phi^{-1}(\mu)$, one can prove, by using (57),(61),(70) and (71), that

$$
\begin{aligned}
&(K \circ \iota_\mu)^* \tau(\xi_P(w), \eta_P(w)) \\
&= \tau(\xi_Q(\nu), \eta_Q(\nu)) = (\nu | [\xi, \eta]) = \tfrac{1}{4}\mathrm{tr}(Gw^*w(\xi\eta - \eta\xi) - (\eta^*\xi^* - \xi^*\eta^*)w^*wG) \\
&= \tfrac{1}{4}\mathrm{tr}(G(w\xi)^*w\eta - G(w\eta)^*w\xi + (w\xi)^*w\eta G - (w\eta)^*w\xi G) \\
&= \tfrac{1}{4}\mathrm{tr}(G\mathrm{d}w^* \wedge \mathrm{d}w + \mathrm{d}w^* \wedge \mathrm{d}wG)(\xi_P(w), \eta_P(w)) = \mathrm{d}\theta(\xi_P(w), \eta_P(w)),
\end{aligned} \tag{72}
$$

where use has been made of the fact that

$$\mathrm{tr}(AB + B^*A^*) = \mathrm{tr}(BA + A^*B^*), \quad A, B \in M(n, \mathbf{H}). \tag{73}$$

Eq. (72) means that

$$(K \circ \iota_\mu)^* \tau = \iota_\mu^* \mathrm{d}\theta. \tag{74}$$

From (39),(69) and (74), it follows that $\pi_\mu^* \sigma_\mu = \pi_\mu^* \tilde{K}_\mu^* \tau$, and therefore, on account of the injectivity of π_μ^*, that

$$\tilde{K}_\mu^* \tau = \sigma_\mu, \tag{75}$$

which means that $\tilde{K}_\mu$ is a symplectic map. This ends the proof. □

3.4. $SO^*(8)$ as a dynamical group

In this subsection, we are to show that the group $Sp(2\mathrm{i}, 2\mathrm{i})_0 \cong SO^*(8)$ should be called a dynamical group for the conformal Kepler problem as well as for the $SU(2)$ Kepler problem. To this end, we deals with the momentum map K in components with respect to the basis of $sp(2\mathrm{i}, 2\mathrm{i})$. For each base $e_n = E_n/2$ given in Appendix A, the component of the momentum map K is denoted by

$$K_n(w) = (K(w) | e_n), \quad n = 1, 2, \ldots, 28. \tag{76}$$

We give the explicit expression of K_n in Appendix C. In particular, for $n = 28, 26$, the components K_n take the form,

$$
\begin{aligned}
K_{28} &= -\tfrac{1}{4}(w_1^*w_1 + w_2^*w_2 + w_3^*w_3 + w_4^*w_4) \\
&= -\tfrac{1}{4}(|q_1|^2 + |q_2|^2 + |p_1|^2 + |p_2|^2),
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
K_{26} &= -\tfrac{1}{4}(\mathrm{i}w_1^*w_3\mathrm{i} + \mathrm{i}w_2^*w_4\mathrm{i} + \mathrm{i}w_3^*w_1\mathrm{i} + \mathrm{i}w_4^*w_2\mathrm{i}) \\
&= -\tfrac{1}{4}(-|q_1|^2 - |q_2|^2 + |p_1|^2 + |p_2|^2).
\end{aligned} \tag{78}
$$

Set $K_{28} = -A_{-1}/2$ and $K_{26} = -A_1/2$. Then one has

$$A_0 := \tfrac{1}{2}(A_1 + A_{-1}) = \tfrac{1}{2}(|p_1|^2 + |p_2|^2), \quad A_{-1} - A_1 = |q_1|^2 + |q_2|^2. \tag{79}$$

A_{-1}, A_1 and A_0 are Hamiltonians for the harmonic oscillator, the repulsive oscillator, and a free particle, respectively. We introduce here the Hamiltonian with a parameter ε,

$$A_\varepsilon = \tfrac{1}{2}(p \cdot p - \varepsilon q \cdot q), \tag{80}$$

which is associated with the harmonic oscillator, the repulsive oscillator, or a free particle, depending on whether ε is negative, positive, or zero. From (79), we observe that A_ε is expressed as a linear combination of components K_n of the momentum map K. From (44) and (80), the A_ε is related to H_C by

$$4r(H_C - \varepsilon) = A_{8\varepsilon} - 4\kappa. \tag{81}$$

Hence, the energy manifold $H_C^{-1}(E)$ of the conformal Kepler problem coincides with the energy manifold $A_{8E}^{-1}(4\kappa)$ for the Hamiltonian A_{8E}. (Strictly speaking, the closure of $H_C^{-1}(E)$ coincides with $A_{8E}^{-1}(4\kappa)$.) Further, the Hamiltonian flow of H_C coincides with that of A_{8E} within a parameter change on the energy manifold; $4rX_{H_C} = X_{A_{8E}}$ on $H_C^{-1}(E)$. Thus, we are allowed to work with A_{8E}, instead of H_C, on $H_C^{-1}(E)$. In view of this, the group $Sp(2\mathrm{i}, 2\mathrm{i})_0 \cong SO^*(8)$ and its Lie algebra $sp(2\mathrm{i}, 2\mathrm{i}) \cong so^*(8)$ should be called a dynamical group and a dynamical Lie algebra of the conformal Kepler problem, respectively. For the sake of simplicity, we refer to $Sp(2\mathrm{i}, 2\mathrm{i})$ as the dynamical group, without restricting to the identity component.

We proceed to show that the group $Sp(2\mathrm{i}, 2\mathrm{i})$ can also be regarded as a dynamical group of the $SU(2)$ Kepler problem. Since $Sp(2\mathrm{i}, 2\mathrm{i})$ and $Sp(1)$ commute, and since $\Phi^{-1}(\mu)$ is invariant under the $Sp(2\mathrm{i}, 2\mathrm{i})$ right action from (65), the $Sp(2\mathrm{i}, 2\mathrm{i})$ action projects to that on P_μ; $[w] \mapsto [w]g = [wg]$, $g \in Sp(2\mathrm{i}, 2\mathrm{i})$, which turns out to preserve the reduced symplectic form σ_μ as a consequence of the fact that $Sp(2\mathrm{i}, 2\mathrm{i})$ leaves $\mathrm{d}\theta$ invariant. Further, when restricted to $\Phi^{-1}(\mu)$, the momentum map $K : T^*(\dot{\mathbf{H}}^2) \to sp(2\mathrm{i}, 2\mathrm{i})$ projects to the map $\tilde{K}_\mu : P_\mu \to sp(2\mathrm{i}, 2\mathrm{i})$ on account of (69). In particular, components of $\tilde{K}_\mu$ become functions on P_μ. Hence the reduced function $A_{\mu,\varepsilon}$ determined by $A_{\mu,\varepsilon} \circ \pi_\mu = A_\varepsilon \circ \iota_\mu$ turns out to be expressed as a linear combination of components of $\tilde{K}_\mu$ as well as A_ε is a linear combination of components of K. Further, for a function F on $T^*(\dot{\mathbf{H}}^2)$ and its reduced function F_μ determined by $F_\mu \circ \pi_\mu = F \circ \iota_\mu$, the respective associated Hamiltonian vector fields X_F and X_{F_μ} are related by $\pi_{\mu *}(X_F(\iota_\mu(w))) = X_{F_\mu}(\pi_\mu(w))$, so that one has $4rX_{H_\mu} = X_{A_{\mu,8E}}$ on the energy manifold $H_\mu^{-1}(E)$ from the similar relation $4rX_{H_C} = X_{A_{8E}}$ on $H_C^{-1}(E)$. Thus $Sp(2\mathrm{i}, 2\mathrm{i})$ can be regarded as a dynamical group for the $SU(2)$ Kepler problem as well. On account of (52) and (60), we then obtain the following.

Theorem 3. *The $SU(2)$ Kepler problem admits a dynamical group $SO^*(8)$.*

4. Symmetry of the $SU(2)$ Kepler problem

A symmetry group of the $SU(2)$ Kepler problem is, of course, a group which leaves $(P_\mu, \sigma_\mu, H_\mu)$ invariant. Since $Sp(2\mathrm{i}, 2\mathrm{i})$ leaves σ_μ invariant, we may consider the symmetry group of the $SU(2)$ Kepler problem as a subgroup of $Sp(2\mathrm{i}, 2\mathrm{i})$ that leaves the Hamiltonian H_μ invariant. In order to study the symmetry group of the $SU(2)$ Kepler problem, however, it is of great help to consider the symmetry group of the conformal Kepler problem and then apply the reduction method. As was stated already, we are allowed to work with A_{8E}, instead of H_C, on the energy manifold $H_C^{-1}(E)$. After the reduction procedure, the energy manifold $H_\mu^{-1}(E)$ of the $SU(2)$ Kepler problem is then given by

$$H_\mu^{-1}(E) = \pi_\mu \left(A_{8E}^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \right). \tag{82}$$

Thus the symmetry groups of the $SU(2)$ Kepler problem are looked upon as those groups the action of which is commutative with the $Sp(1)$ action and preserves both of the Hamiltonian A_{8E} and the momentum Φ. Since the action of $Sp(2\mathrm{i}, 2\mathrm{i})$ and of $Sp(1)$ commute, we may consider symmetry groups of the $SU(2)$ Kepler problem as subgroups of $Sp(2\mathrm{i}, 2\mathrm{i})$ that preserve A_{8E} and Φ. Since the topology of the respective energy manifolds depends only on the sign of E, it is convenient to set $8E = -1, 1$, or 0. In what follows, we are to work with the group $GJ(4, \mathbf{H})$, instead of $Sp(2\mathrm{i}, 2\mathrm{i})$. At the same time, we use Ψ for the momentum map, instead of Φ.

4.1. The symmetry group for negative energies

The symmetry group in the case of negative energies was already studied in the previous paper [6]. However, we here treat it again for the sake of consistency. The symmetry group for $E < 0$ is a subgroup of $GJ(4, \mathbf{H})$ which preserves both of A_{-1} and Ψ. Since $GJ(4, \mathbf{H})$ already preserves Ψ, as is observed from (59), we have only to seek for a subgroup that preserves A_{-1}. To this end, we look for a subgroup which commutes with the one-parameter subgroup generated by the Hamiltonian vector field $X_{A_{-1}}$,

$$\exp tX_{A_{-1}} = \begin{pmatrix} E_0 \cos t & -E_0 \sin t \\ E_0 \sin t & E_0 \cos t \end{pmatrix}. \tag{83}$$

A simple calculation shows the subgroup we want is put in the form

$$G^- := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix};\ ab^* = ba^*,\ \ aa^* + bb^* = E_0,\ \ a, b \in M(2, \mathbf{H}) \right\}. \tag{84}$$

Applying the algebra isomorphism $\rho : M(4, \mathbf{H}) \to MJ(8, \mathbf{C})$, one has, for an element of G^-,

$$\rho \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{85}$$

where $\rho(a) = A$ and $\rho(b) = B$ with $A,\ B \in MJ(4, \mathbf{C})$. From (84), one has

$$\rho(G^-) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix};\ AB^* = BA^*,\ \ AA^* + BB^* = I_4,\ \ A, B \in MJ(4, \mathbf{C}) \right\}, \tag{86}$$

with I_4 the 4×4 identity matrix. A simple calculation shows that

$$\tilde{S}^{-1} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tilde{S} = \begin{pmatrix} A + \mathrm{i}B & 0 \\ 0 & A - \mathrm{i}B \end{pmatrix}, \quad \tilde{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_4 & I_4 \\ \mathrm{i}I_4 & -\mathrm{i}I_4 \end{pmatrix}, \tag{87}$$

where $\tilde{S}$ is an extension of the matrix S defined in (54). We note here that $MJ(4, \mathbf{C}) + \mathrm{i}MJ(4, \mathbf{C}) = M(4, \mathbf{C})$. Further, from the condition (48) to which A and B are subject, it follows that $A + \mathrm{i}B \in U(4)$ and $A - \mathrm{i}B = J_d{}^{\mathrm{t}}(A + \mathrm{i}B)^* J_d^{-1}$. Thus, it turns out that $G^- \cong U(4)$. The one-parameter group given by (83) is also a subgroup of G^-. Factoring out this subgroup, one obtains $U(4)/U(1) \cong SU(4)$, which will give a symmetry group of the $SU(2)$ Kepler problem for negative energies.

4.2. The symmetry group for positive energies

In the case of $E > 0$, the symmetry group should preserve A_1 and Ψ. As in the case of $E < 0$, we have only to look for a subgroup of $GJ(4, \mathbf{H})$ which commutes with the one-parameter group generated by X_{A_1},

$$\exp tX_{A_1} = \begin{pmatrix} E_0 \cosh t & E_0 \sinh t \\ E_0 \sinh t & E_0 \cosh t \end{pmatrix}. \tag{88}$$

Hence, we find the subgroup

$$G^+ := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix};\ ab^* = ba^*,\ \ aa^* - bb^* = E_0,\ \ a, b \in M(2, \mathbf{H}) \right\}. \tag{89}$$

The application of the algebra isomorphism $\rho : M(4, \mathbf{H}) \to J(8, \mathbf{C})$ provides

$$\rho(G^+) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix};\ AB^* = BA^*,\ \ AA^* - BB^* = I_4,\ \ A, B \in MJ(4, \mathbf{C}) \right\}. \tag{90}$$

Like (87), one has

$$T^{-1} \begin{pmatrix} A & B \\ B & A \end{pmatrix} T = \begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix}, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_4 & I_4 \\ I_4 & -I_4 \end{pmatrix}. \tag{91}$$

We note further that $(A + B)J_d = J_d(\overline{A + B})$ and $A - B = ((A + B)^{-1})^*$. Thus $\rho(G^+)$ turns out to be isomorphic to $MJ(4, \mathbf{C})$ defined already in (48). The one-parameter group (88) is a subgroup of G^+. Factoring out this subgroup, one has

$$SU^*(4) = \{g \in M(4, \mathbf{C});\ \ gJ_d = J_d\bar{g},\ \ \det g = 1\}, \tag{92}$$

which will yield a symmetry group of the $SU(2)$ Kepler problem for positive energies.

4.3. The symmetry group for the zero energy

For $E = 0$, the symmetry group has to preserve A_0 and Ψ, and therefore, should commute with the one-parameter group

$$\exp tX_{A_0} = \begin{pmatrix} E_0 & 0 \\ tE_0 & E_0 \end{pmatrix}. \tag{93}$$

A calculation gives a subgroup of $GJ(4, \mathbf{H})$ which is commutative with (93)

$$G^0 = \left\{ \begin{pmatrix} a & 0 \\ c & a \end{pmatrix};\ aa^* = E_0,\ ac^* = ca^*,\ a, c \in M(2, \mathbf{H}) \right\}. \tag{94}$$

Setting $M = ac^*$, one finds that $M = M^*$ and $c = Ma$. Then one has

$$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ Ma & a \end{pmatrix} = \begin{pmatrix} E_0 & 0 \\ M & E_0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}. \tag{95}$$

It follows from this that for arbitrary elements (a, M), (b, N) with $a, b \in Sp(2)$ and $M, N \in \mathbf{R}^6$, where $\mathbf{R}^6$ stands for the set of quaternionic Hermitian 2×2 matrices, the product of them is expressed as

$$(a, M) \cdot (b, N) = (ab, M + aNa^*). \tag{96}$$

This implies that G^0 is broken up into a semi-direct product

$$G^0 \cong Sp(2) \times_S \mathbf{R}^6. \tag{97}$$

The one-parameter group (93) is a subgroup of G^0. If this subgroup is got rid of, one obtains the group $Sp(2) \times_S \mathbf{R}^5$, which may be viewed as a symmetry group for the $SU(2)$ Kepler problem of the zero energy, where $\mathbf{R}^5$ stands for the set of quaternionic Hermitian 2×2 matrices with vanishing trace.

4.4. A summary of symmetry groups

So far we have obtained the subgroups $SU(4)$, $SU^*(4)$, and $Sp(2) \times_S \mathbf{R}^5$ of the dynamical group $GJ(4, \mathbf{H})_0 \cong SO^*(8)$ of the conformal Kepler problem. Since the respective energy manifolds $H_C^{-1}(E)$ are viewed as submanifolds of $\mathcal{O}_{K(p_0)}$, the action of those subgroups is considered as the co-adjoint action, so that g and $-g$ have the same action on the energy manifolds. Therefore, those subgroups should project, respectively, to $SU(4)/\mathbf{Z}_2 \cong SO(6)$, $SU^*(4)/\mathbf{Z}_2 \cong O_0(5, 1)$, and $(Sp(2) \times_S \mathbf{R}^5)/\mathbf{Z}_2 \cong SO(5) \times_S \mathbf{R}^5$ to act on the energy manifold $H_\mu^{-1}(E)$ of the $SU(2)$ Kepler problem, where $O_0(5, 1)$ denotes the identity component of $O(5, 1)$. See [30] for the proof of $SU(4)/\mathbf{Z}_2 \cong SO(6)$, $SU^*(4)/\mathbf{Z}_2 \cong O_0(5, 1)$, and $Sp(2)/\mathbf{Z}_2 \cong SO(5)$. In the case of $(Sp(2) \times_S \mathbf{R}^5)/\mathbf{Z}_2$, $(-a, M) \in Sp(2) \times_S \mathbf{R}^5$ is identified with (a, M), and $\mathbf{Z}_2 = \{I_2, -I_2\}$. Thus we have the following theorem on symmetry groups.

Theorem 4. *The symmetry group for the $SU(2)$ Kepler problem is isomorphic with $SO(6)$, $O_0(5, 1)$, or $SO(5) \times_S \mathbf{R}^5$, according to whether the energy is negative, positive, or zero.*

We here have to make a remark on the kinematical symmetry of the $SU(2)$ Kepler problem. The intersection of G^+, G^-, and G^0 is given by

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix};\ aa^* = E_0,\ a \in M(2, \mathbf{H}) \right\} \cong Sp(2). \tag{98}$$

This group acts on the phase space $\dot{\mathbf{H}}^2 \times \mathbf{H}^2$ in the manner

$$(q, p) \mapsto (qa, pa), \quad a \in Sp(2). \tag{99}$$

Since $(q \cdot q)^2 = \sum_\ell u_\ell^2$ and $p \cdot p = 4r\sum_\ell v_\ell^2 + \sum_\alpha \Psi_\alpha^2/r$, and since the $Sp(2)$ action leaves $q \cdot q$ and $p \cdot p$ invariant, the $Sp(2)$ action project to the action on the reduced phase space P_μ, preserving $\sum_\ell u_\ell^2$ and $\sum_\ell v_\ell^2$. It then follows that the $Sp(2)$ action project to the $SO(5)$ action on the reduced phase space, which is interpreted as the kinematical symmetry of the $SU(2)$ Kepler problem. Note also that $SO(5)$ is the intersection among the symmetry groups $SO(6)$, $O_0(5,1)$, and $SO(5) \times_S \mathbf{R}^5$.

4.5. Another subgroup of the dynamical group

In Section 3, we have treated the components, K_{28} and K_{26}, of the momentum map K associated with the dynamical group. Those are associated with the harmonic and repulsive oscillators, respectively. Here we consider the component K_{27} given in Appendix C,

$$K_{27} = \frac{1}{2}\sum_{m=0}^{7} x_m y_m = \frac{1}{2} q \cdot p. \tag{100}$$

The one-parameter group generated by the Hamiltonian vector field associated with K_{27} is given by

$$\exp tX_{K_{27}} = \begin{pmatrix} e^{t/2}E_0 & 0 \\ 0 & e^{-t/2}E_0 \end{pmatrix}. \tag{101}$$

In the preceding subsections, we have sought for subgroups of the dynamical group $GJ(4, \mathbf{H})$ which are commutative with the respective one-parameter subgroups $\exp tX_{A_{-1}}$, etc. We here look for the subgroup which is commutative with $\exp tX_{K_{27}}$ given above. A simple calculation provides the subgroup

$$G^a = \left\{ \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^* \end{pmatrix}, \quad a \in GL(2, \mathbf{H}) \right\}. \tag{102}$$

Applying the map $\rho : \ M(4, \mathbf{H}) \to MJ(8, \mathbf{C})$, and imposing the unimodular condition, we obtain

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}, \quad A \in MJ(4, \mathbf{C}), \ \det A = 1 \right\} \cong SU^*(4). \tag{103}$$

5. Constants of motion

In this section, we are to find constants of motion of the $SU(2)$ Kepler problem, which form, under the Poisson bracket, the Lie algebras of the symmetry groups discussed already in Section 4.5. To start with, we wish to express the Poisson brackets among the variables

u_ℓ and v_ℓ introduced in Section 2. To this end, we introduce vector fields $(\partial/\partial u_\ell)^*$ and the curvature of the connection treated in Section 2, which are, respectively, defined by

$$\omega_\alpha\left((\partial/\partial u_\ell)^*\right) = 0, \quad \mathrm{d}u_\ell\left((\partial/\partial u_m)^*\right) = \delta_{\ell m}, \tag{104}$$

and by

$$\Omega_\alpha := \mathrm{d}\omega_\alpha - \sum_{\beta<\gamma} \varepsilon_{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma. \tag{105}$$

The $(\partial/\partial u_\ell)^*$ and Ω_α prove to be expressed as

$$\left(\frac{\partial}{\partial u_\ell}\right)^* = \frac{1}{2r}H_\ell, \quad \Omega_\alpha = \sum_{\ell<m} F^\alpha_{\ell m}\,\mathrm{d}u_\ell \wedge \mathrm{d}u_m, \tag{106}$$

respectively, where H_ℓ are the horizontal vector fields given in (18), and $F^\alpha_{\ell m}$ are components of Ω_α. Further, by definition, one has

$$\Omega\left(\left(\frac{\partial}{\partial u_\ell}\right)^*, \left(\frac{\partial}{\partial u_m}\right)^*\right) = -\omega\left(\left[\left(\frac{\partial}{\partial u_\ell}\right)^*, \left(\frac{\partial}{\partial u_m}\right)^*\right]\right). \tag{107}$$

A simple calculation then provides

$$\left[\left(\frac{\partial}{\partial u_\ell}\right)^*, \left(\frac{\partial}{\partial u_m}\right)^*\right] = -\sum_\alpha F^\alpha_{\ell m} V_\alpha, \tag{108}$$

where use has been made of the fact that $\omega_\alpha(V_\beta) = \delta_{\alpha\beta}$ and that $\mathrm{d}u_\ell((\partial/\partial u_m)^*) = \delta_{\ell m}$. Now we are in a position to calculate the Poisson brackets among u_ℓ, v_ℓ. A calculation results in

$$\{u_\ell, u_m\} = 0, \quad \{u_\ell, v_m\} = \delta_{\ell m}, \quad \{v_\ell, v_m\} = \sum_{\alpha=1}^{3} \Psi_\alpha F^\alpha_{\ell m}. \tag{109}$$

The first two of the above equations are easy to prove. We here give the proof of the last one. Using (108) and the fact that

$$\left(\frac{\partial}{\partial u_\ell}\right)^* (p\cdot q) = v_\ell, \quad V_\alpha(q\cdot p) = \Psi_\alpha, \tag{110}$$

we carry out the calculation to give

$$\begin{aligned}\{v_\ell, v_m\} &= \sum_{n=0}^{7}\left(\frac{\partial v_\ell}{\partial x_n}\frac{\partial v_m}{\partial y_n} - \frac{\partial v_\ell}{\partial y_n}\frac{\partial v_m}{\partial x_n}\right) = \left(\frac{\partial}{\partial u_m}\right)^* v_\ell - \left(\frac{\partial}{\partial u_\ell}\right)^* v_m \\ &= \left[\left(\frac{\partial}{\partial u_m}\right)^*, \left(\frac{\partial}{\partial u_\ell}\right)^*\right](p\cdot q) = -\sum_\alpha F^\alpha_{\ell m} V_\alpha (p\cdot q) = \sum_\alpha \Psi_\alpha F^\alpha_{\ell m}.\end{aligned} \tag{111}$$

Further, as a consequence of (109), we obtain

$$\{r,\ v_\ell\} = u_\ell/r. \tag{112}$$

In addition, a straightforward calculation yields

$$\{u_\ell, \Psi_\alpha\} = \{v_\ell, \Psi_\alpha\} = 0, \quad \{\Psi_\alpha, \Psi_\beta\} = 2\sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma}\Psi_\gamma. \tag{113}$$

We proceed to the Poisson brackets among components of the momentum map K. To make calculation easy to perform, we use the complex numbers instead of the quaternions. Let $\tilde{w} \in \mathbf{C}^8$ and $\tilde{\xi} \in MJ(8, \mathbf{C})$ denote the complex vector and the complex matrix corresponding to $w \in \mathbf{H}^4$ and to $\xi \in M(4, \mathbf{H})$, respectively. Then components K_n given by (76) are expressed as

$$K_n(w) = \tfrac{1}{2}\mathrm{tr}(\tilde{w}^* \tilde{w} \tilde{e}_n \tilde{G}), \tag{114}$$

where we have to be reminded that $\tilde{w}$ and $\tilde{w}^*$ denote row and column vectors, respectively. Further, the symplectic form $\mathrm{d}\theta$ is put in the form

$$\mathrm{d}\theta = -\tfrac{1}{2}\,\mathrm{d}\tilde{w} \wedge \tilde{G}\,\mathrm{d}\tilde{w}^*. \tag{115}$$

Then the Poisson brackets among K_ns turn out to take the form

$$-\{K_n, K_m\} = X_{K_n}(K_m) = \tfrac{1}{2}\mathrm{tr}(\tilde{w}^* \tilde{w} [\tilde{e}_n, \tilde{e}_m] \tilde{G}). \tag{116}$$

Since the (covering groups of) symmetry groups are subgroups of the dynamical group, the constants of motions may be found among those functions K_n that are associated with the respective symmetry groups. We treat first the Lie algebras of the symmetry groups and then proceed to constants of motion. In the case of negative energies, it follows from (84) that the Lie algebra of the group G^- takes the form

$$\mathcal{G}^- = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix};\ \alpha + \alpha^* = 0,\ \beta^* = \beta,\ \alpha, \beta \in M(2, \mathbf{H}) \right\}. \tag{117}$$

The corresponding subalgebra in $sp(2\mathrm{i}, 2\mathrm{i})$ is put in the form $S^{-1}\mathcal{G}^- S$. Writing out this, we find that the matrices e_n with $n = 1, \ldots, 15$ and $n = 28$ form a basis of $S^{-1}\mathcal{G}^- S$. Among these matrices, e_n with $n = 1, \ldots, 15$ form a basis of the symmetry Lie algebra $su(4) \cong so(6)$. Note that e_{28} is commutative with the others.

We turn to the Lie algebra of the symmetry group for positive energies. We observe from (89) that the Lie algebra of G^+ has the form

$$\mathcal{G}^+ = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix};\ \alpha + \alpha^* = 0,\ \beta^* = \beta,\ \alpha, \beta \in M(2, \mathbf{H}) \right\}. \tag{118}$$

Writing out the corresponding subalgebra $S^{-1}\mathcal{G}^+ S$ in $sp(2\mathrm{i}, 2\mathrm{i})$, we see that e_n with $n = 1, \ldots, 10$, $n = 21, \ldots, 25$, and $n = 26$ form a basis of $S^{-1}\mathcal{G}^+ S$. The basis e_n other than e_{26} form a basis of the symmetry Lie algebra $su^*(4) \cong so(1, 5)$. Note also that e_{26} is commutative with the basis of $su^*(4)$.

In the case of the zero energy, the Lie algebra of the group G^0 is expressed as

$$\mathcal{G}^0 = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha \end{pmatrix};\ \alpha + \alpha^* = 0,\ \gamma^* = \gamma,\ \alpha, \gamma \in M(2, \mathbf{H}) \right\}. \tag{119}$$

The corresponding subalgebra in $sp(2\mathrm{i}, 2\mathrm{i})$ is put in the form $S^{-1}\mathcal{G}^0 S$, which shows that a basis of $S^{-1}\mathcal{G}^0 S$ consists of e_n with $n = 1, \dots, 10$ and $e_{11}+e_{24}, e_{14}+e_{21}, e_{15}+e_{23}, e_{12}+e_{22}, e_{13}+e_{25}, e_{26}+e_{28}$. The base matrices other than $e_{26}+e_{28}$ form a basis of the symmetry Lie algebra $sp(2) \oplus_S \mathbf{R}^5 \cong so(5) \oplus_S \mathbf{R}^5$, where $\oplus_S$ denotes a semi-direct sum.

In the same manner as above, we can show that the Lie algebra $\mathcal{G}^a$ of the group G^a given in (102) is put in the form

$$\mathcal{G}^a = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^* \end{pmatrix};\ \alpha \in M(2, \mathbf{H}) \right\}. \tag{120}$$

The corresponding Lie subalgebra $S^{-1}\mathcal{G}^a S$ in $sp(2\mathrm{i}, 2\mathrm{i})$ has a basis consisting of e_n with $n = 1, \dots, 10$, $n = 16, \dots, 20$, and $n = 27$. The base matrices other than e_{27} form a basis of the Lie algebra $su^*(4)$ on account of (103). We have to notice that any of the basis e_n, $n = 1, \dots, 28$, belongs to one of the Lie algebras $\mathcal{G}^-$, $\mathcal{G}^+$, $\mathcal{G}^0$, and $\mathcal{G}^a$. Further, the intersection of the symmetry algebras $su(4)$, $su^*(4)$, and $so(5) \oplus_S \mathbf{R}^5$ has the basis e_n with $n = 1, \dots, 10$. This Lie algebra should be called the kinematical symmetry algebra and is isomorphic with $sp(2) \cong so(5)$ on account of (98).

We are now in a position to obtain constants of motion. We first discuss the constants of motion associated with the kinematical symmetry algebra, which should be called the angular momentum. Looking into the commutation relation table given in Appendix B, we observe that the commutation relations among $e_{16}, \dots, e_{20}$ yield $e_1, \dots, e_{10}$, the basis of the kinematical symmetry algebra. Moreover, the associated functions $K_{16}, \dots, K_{20}$, which are given in Appendix C, can be shown to be expressed as

$$K_{16} = rv_4, \quad K_{17} = rv_0, \quad K_{18} = rv_1, \quad K_{19} = rv_3, \quad K_{20} = rv_2. \tag{121}$$

In view of these facts, we get interested in the Poisson brackets among $K_{16}, \dots, K_{20}$. Each of the resultant brackets is equal to one of $K_1, \dots, K_{10}$. Let $L_{\ell m} = -\{rv_\ell, rv_m\}$. Then a calculation together with (109) and (112) results in

$$L_{\ell m} := -\{rv_\ell, rv_m\} = u_\ell v_m - u_m v_\ell - r^2 \sum_{\alpha=1}^{3} \Psi_\alpha F^\alpha_{\ell m}, \quad \ell, m = 0, \dots, 4. \tag{122}$$

The $L_{\ell m}$ are constants of motion for the conformal Kepler problem. In fact, since $K_1, \dots, K_{10}$ commute with K_{26}, K_{28} and hence with $A_{8\varepsilon}$, and since $\{K_n, r\} = 0$ for $n = 1, \dots, 10$, one verifies that $\{K_n, H_\mathrm{C}\} = 0$, $n = 1, \dots, 10$, on account of (81), and hence that $\{L_{\ell m}, H_\mathrm{C}\} = 0$.

We proceed to functions other than $K_1, \dots, K_{10}$. We set $D_\ell^+ = 1/2\{K_{26}, rv_\ell\}$ and $D_\ell^- = 1/2\{K_{28}, rv_\ell\}$. Then a straightforward calculation provides

$$\begin{aligned} D_\ell^+ &:= \tfrac{1}{2}\{K_{26}, rv_\ell\} = \tfrac{1}{8}(H'_\ell \cdot p + u_\ell), \\ D_\ell^- &:= \tfrac{1}{2}\{K_{28}, rv_\ell\} = \tfrac{1}{8}(H'_\ell \cdot p - u_\ell), \end{aligned} \qquad \ell = 0, \dots, 4, \tag{123}$$

where H'_ℓs denote vectors which are obtained from H_ℓ by replacing $y_m, m = 0, \dots, 7$ for x_m. The commutation relation table shows that D_ℓ^+ and D_ℓ^- are equal to one of $K_{11}, \dots, K_{15}$

and to one of $K_{21}, \dots, K_{25}$, respectively, up to $\pm$ sign and the factor 1/2. Hence, one further obtains $\{D_\ell^+, K_{28}\} = 0$ and $\{D_\ell^-, K_{26}\} = 0$ from the same table. From these brackets together with (81), it follows that

$$\left.\begin{aligned}\{D_\ell^+, H_C\} &= -\tfrac{1}{r}\left(H_C + \tfrac{1}{8}\right)\{D_\ell^+, r\},\\ \{D_\ell^-, H_C\} &= -\tfrac{1}{r}\left(H_C - \tfrac{1}{8}\right)\{D_\ell^-, r\},\end{aligned}\right\} \quad \ell = 0, \dots, 4. \tag{124}$$

These equations imply that D_ℓ^+ and D_ℓ^- are constants of motion for the conformal Kepler problem, if restricted on $H_C^{-1}(-1/8)$ and on $H_C^{-1}(1/8)$, respectively. It follows further that $D_\ell^+ + D_\ell^- = \frac{1}{4}H'_\ell \cdot p$ is also a constant of motion, if restricted on $H_C^{-1}(0)$. In view of these facts, we define functions

$$D_\ell := \tfrac{1}{8}(H'_\ell \cdot p - 8u_\ell H_C), \quad \ell = 0, \dots, 4. \tag{125}$$

Now we can show that D_ℓ are constants of motion. In fact, expanding the bracket $(1/8)\{H'_\ell \cdot p - 8u_\ell H_C, H_C\}$, and using $u_\ell = 4(D_\ell^+ - D_\ell^-)$ and $H'_\ell \cdot p = 4(D_\ell^+ + D_\ell^-)$ together with (124), we obtain $\{D_\ell, H_C\} = 0$. Note also that $\{D_\ell^+, r\} = \{D_\ell^-, r\}$ thanks to $\{u_\ell, r\} = 0$. Writing out the right-hand side of (125) along with $H'_\ell \cdot p = r\{v_\ell, p \cdot p\} + 4v_\ell q \cdot p$, etc., we obtain

$$D_\ell = \frac{1}{2}r^2\{v_\ell, \sum v_m^2\} - u_\ell \sum v_m^2 + v_\ell \sum u_m v_m + \frac{\kappa}{r}u_\ell = \sum_{m=0}^{4} L_{m\ell} v_m + \frac{\kappa}{r}u_\ell. \tag{126}$$

The commutation relations among $L_{\ell m}$, D_ℓ are calculated to give

$$\begin{aligned}&\{L_{\ell m}, L_{mn}\} = -L_{\ell n}, \quad \{L_{\ell m}, L_{nh}\} = 0, \quad (\neq \ell, m, n, h),\\ &\{L_{\ell m}, D_n\} = \delta_{\ell n} D_m - \delta_{mn} D_\ell, \quad \{D_\ell, D_m\} = -2H_C L_{\ell m}.\end{aligned} \tag{127}$$

So far we have studied the constants of motion for the conformal Kepler problem, which are summarized as follows.

Proposition 5. *The conformal Kepler problem has the constants of motion $L_{\ell m}$ and D_ℓ which are given in (122) and (126), respectively. The commutation relations among them are given in (127).*

Since these constants of motion are $Sp(1)$-invariant, they project to functions on the reduced phase space P_μ. The reduced constants of motion are indeed determined through

$$[L_{\ell m}]_\mu \circ \pi_\mu = L_{\ell m} \circ \iota_\mu, \quad [D_\ell]_\mu \circ \pi_\mu = D_\ell \circ \iota_\mu, \tag{128}$$

and expressed, on account of (122) and (126), as

$$[L_{\ell m}]_\mu = u_\ell v_m - u_m v_\ell - r^2 \sum_{\alpha=1}^{3} \mu_\alpha F_{\ell m}^\alpha, \qquad [D_\ell]_\mu = \sum_{m=0}^{4} [L_{m\ell}]_\mu v_m + \frac{\kappa}{r} u_\ell. \tag{129}$$

Here $\sum_\alpha \mu_\alpha F_{\ell m}^\alpha$ should be viewed as functions on the reduced phase space. In fact, from the transformation property $\Omega_{hq} = \mathrm{Ad}_h \Omega_q$, $h \in Sp(1)$, a consequence of (20), and from the definition of G_μ, one verifies that $\sum_{\ell<m}\sum_\alpha \mu_\alpha F^\alpha_{\ell m} \mathrm{d}u^\ell \wedge \mathrm{d}u^m = \langle\mu|\Omega\rangle$ is invariant under the action of G_μ. Since $L_{\ell m}$ are associated with the kinematical symmetry algebra $sp(2) \cong so(5)$, and since the $Sp(2)$ action projects to the $SO(5)$ action on the reduced phase space, i.e., on the phase space of the $SU(2)$ Kepler problem, the reduced functions $[L_{\ell m}]_\mu$ are interpreted as the angular momenta of the $SU(2)$ Kepler problem. Moreover, the reduced functions $[D_\ell]_\mu$ are looked upon as the Runge–Lenz vectors.

Since the Poisson bracket relations among $Sp(1)$-invariant functions project to the same Poisson brackets relations among reduced functions, the $[L_{\ell m}]_\mu$ and $[D_\ell]_\mu$ become also constants of motion for the $SU(2)$ Kepler problem, and subject to the same Poisson bracket relation as (127),

$$\begin{aligned}&\{[L_{\ell m}]_\mu, [L_{mn}]_\mu\} = -[L_{\ell n}]_\mu, \quad \{[L_{\ell m}]_\mu, [L_{nh}]_\mu\} = 0, \quad (\neq \ell, m, n, h),\\ &\{[L_{\ell m}]_\mu, [D_n]_\mu\} = \delta_{\ell n}[D_m]_\mu - \delta_{mn}[D_\ell]_\mu, \{[D_\ell]_\mu, [D_m]_\mu\} = -2H_\mu[L_{\ell m}]_\mu,\end{aligned} \tag{130}$$

where we have to notice that the Poisson brackets for the reduced functions are, of course, defined through the reduced symplectic form σ_μ.

Theorem 6. *The $SU(2)$ Kepler problem $(P_\mu, \sigma_\mu, H_\mu)$ has the constants of motion $[L_{\ell m}]_\mu$ and $[D_\ell]_\mu$, which are given in (129). The Poisson brackets among them are given in (130), which shows that according to whether the energy H_μ is fixed negative, positive, or zero, the symmetry Lie algebra is isomorphic to $so(6)$, $so(1,5)$, or $e(5)$, respectively, where $e(5) \cong so(5) \oplus_S \mathbf{R}^5$ denotes the Lie algebra of the group of Euclid motions in $\mathbf{R}^5$. All the symmetry Lie algebras are subalgebras of the dynamical Lie algebra $so^*(8) \cong so(2,6)$.*

We remark here that we have already shown in [29] that a theorem similar to the above holds in the quantized $SU(2)$ Kepler problem. The constants of motion $[L_{\ell m}]_\mu$ and $[D_\ell]_\mu$ have their corresponding operators with μ quantized to take discrete values. It is also shown that the quantized $SU(2)$ Kepler problem has the dynamical Lie algebra $so(2,6)$ [31].

6. Concluding remarks

In the case of the MIC–Kepler problem [20], we treated the group $U(1) \times SU(2,2)$ and showed that the phase space of the MIC–Kepler problem is realized as a co-adjoint orbit of the dynamical group $SU(2,2)$. The same situation occurs in the $SU(2)$ Kepler problem; as was already pointed out in Section 3, the full symplectic group we are concerned with is $Sp(1) \times GJ(4, \mathbf{H})$ or $Sp(1) \times SO^*(8)$, and the phase space of the $SU(2)$ Kepler problem is a co-adjoint orbit of the dynamical group $SO^*(8)$. Each of these co-adjoint orbits is a specialization of a more general theorem in part. As was stated in [20], the general setting for co-adjoint orbits is given by the following theorem of Kummer [11,12].

Theorem 7. *Let (M, σ) be a Hamiltonian $G = K \times H$-space, where σ is a symplectic form on M, and K and H are connected Lie groups. Let $\phi_K \oplus \phi_H : M \to \mathcal{K}^* \oplus \mathcal{H}^*$ be the equivariant momentum maps associated with the $K \times H$ action, where $\mathcal{K}$ and $\mathcal{H}$ are the Lie algebras of K and H, respectively, and $\mathcal{K}^*$ and $\mathcal{H}^*$ are their duals. Assume that $\mu \in \mathcal{K}^*$*

is a regular value of ϕ_K *, and the reduced phase space* $\phi_K^{-1}(\mu)/K_\mu$ *is a manifold, where* K_μ *is the isotropy subgroup of K at* μ. *Assume further that* $K_\mu \times H$ *acts transitively on* $\phi_K^{-1}(\mu)$. *Then* $\phi_H(\phi_K^{-1}(\mu))$ *is a co-adjoint orbit* $\mathcal{O}$ *of H, and* $\phi_H : \phi_K^{-1}(\mu) \to \mathcal{O}$ *projects to the quotient to define a symplectic covering map* $\tilde{\phi}_H : \phi_K^{-1}(\mu)/K_\mu \to \mathcal{O}$, *where* $\mathcal{O}$ *is considered as a symplectic manifold equipped with the KKS form.*

For co-adjoint orbit structure of the phase space, the 'regularized' Kepler problem was already generalized to the n-dimensional case; according to [32,33], the phase space of the 'regularized' n-dimensional Kepler problem is a co-adjoint orbit of $SO(2, n+1)$. If $\mu = 0$, Theorem 2 falls under this proposition with $n = 5$.

As to the groups $SO^*(8)$ and $SO(2, 6)$, Yokota and Miyashita [30] claim that there are no epimorphisms between $SO^*(8)$ and $SO(2, 6)$, though their Lie algebras are isomorphic to each other.

Appendix A

A basis of $sp(2i, 2i)$.

$$E_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad E_2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & j & 0 \\ 0 & 0 & 0 & j \\ j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \\ k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \end{pmatrix},$$

$$E_7 = \begin{pmatrix} 0 & 0 & j & 0 \\ 0 & 0 & 0 & -j \\ j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \end{pmatrix}, \quad E_8 = \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & -k \\ k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 0 & 0 & 0 & j \\ 0 & 0 & j & 0 \\ 0 & j & 0 & 0 \\ j & 0 & 0 & 0 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & 0 & 0 & k \\ 0 & 0 & k & 0 \\ 0 & k & 0 & 0 \\ k & 0 & 0 & 0 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 0 & j \\ 0 & 0 & -j & 0 \\ 0 & -j & 0 & 0 \\ j & 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

$$E_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad E_{15} = \begin{pmatrix} 0 & 0 & 0 & k \\ 0 & 0 & -k & 0 \\ 0 & -k & 0 & 0 \\ k & 0 & 0 & 0 \end{pmatrix},$$

$$E_{16} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad E_{17} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{18} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad E_{19} = \begin{pmatrix} 0 & k & 0 & 0 \\ -k & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & -k & 0 \end{pmatrix},$$

$$E_{20} = \begin{pmatrix} 0 & j & 0 & 0 \\ -j & 0 & 0 & 0 \\ 0 & 0 & 0 & j \\ 0 & 0 & -j & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{22} = \begin{pmatrix} 0 & j & 0 & 0 \\ -j & 0 & 0 & 0 \\ 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & k & 0 & 0 \\ -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k \\ 0 & 0 & k & 0 \end{pmatrix},$$

$$E_{24} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad E_{25} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$E_{26} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, E_{27} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$E_{28} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Appendix B

The commutation relations among the basis $e_n := E_n/2$ of $sp(2\mathrm{i}, 2\mathrm{i})$.

Here the positive numbers n in the Table B.1 denote the subscript of the basis e_n and 0 means that the corresponding commutation relations vanish. The minus signs in the table means that the resultant vector is inverted; $-n$ denotes $-e_n$.

Table B.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	0	0	0	6	−5	8	−7	10	−9	0	15	0	0	−12	0	0	0	−20	19	0	23	−22	0	0	0	0	0
2	0	0	−4	3	8	−7	6	−5	0	0	0	0	−14	13	0	0	18	−17	0	0	25	0	0	0	−21	0	0	0
3	0	4	0	−2	10	−9	0	0	6	−5	14	0	0	−11	0	−18	0	16	0	0	−24	0	0	21	0	0	0	0
4	0	−3	2	0	0	0	−9	−10	7	8	−13	0	11	0	0	−17	16	0	0	0	0	0	0	−25	24	0	0	0
5	−6	−8	−10	0	0	1	0	2	0	3	0	14	0	−12	0	0	0	−19	18	0	−22	21	0	0	0	0	0	0
6	5	7	9	0	−1	0	−2	0	−3	0	0	0	0	−15	14	0	0	20	0	−18	−23	0	21	0	0	0	0	0
7	−8	−6	0	9	0	2	0	1	−4	0	0	0	15	0	−13	0	20	0	0	−17	0	0	−25	0	23	0	0	0
8	7	5	0	10	−2	0	−1	0	0	−4	0	13	−12	0	0	0	19	0	−17	0	0	25	0	0	−22	0	0	0
9	−10	0	−6	−7	0	3	4	0	0	1	−15	0	0	0	11	−20	0	0	0	16	0	0	24	−23	0	0	0	0
10	9	0	5	−8	−3	0	0	4	−1	0	12	−11	0	0	0	−19	0	0	16	0	0	−24	0	22	0	0	0	0
11	0	0	−14	13	0	0	0	0	15	−12	0	10	−4	3	−9	26	0	0	0	0	0	0	0	−27	0	−16	24	0
12	−15	0	0	0	−14	0	0	−13	0	11	−10	0	8	5	1	0	0	0	−26	0	0	−27	0	0	0	19	22	0
13	0	14	0	−11	0	0	−15	12	0	0	4	−8	0	−2	7	0	26	0	0	0	0	0	0	0	−27	−17	25	0
14	0	−13	11	0	12	15	0	0	0	0	−3	−5	2	0	−6	0	0	−26	0	0	−27	0	0	0	0	18	21	0
15	12	0	0	0	0	−14	13	0	−11	0	9	−1	−7	6	0	0	0	0	0	26	0	0	−27	0	0	−20	23	0
16	0	0	18	17	0	0	0	0	20	19	−26	0	0	0	0	0	4	3	10	9	0	0	0	−28	0	−11	0	−24
17	0	−18	0	−16	0	0	−20	−19	0	0	0	0	−26	0	0	−4	0	−2	−8	−7	0	0	0	0	−28	−13	0	−25
18	0	17	−16	0	19	−20	0	0	0	0	0	0	0	26	0	−3	2	0	5	−6	28	0	0	0	0	14	0	21
19	20	0	0	0	−18	0	0	17	0	−16	0	26	0	0	0	−10	8	−5	0	1	0	28	0	0	0	12	0	22
20	−19	0	0	0	0	18	17	0	−16	0	0	0	0	0	−26	−9	7	6	−1	0	0	0	−28	0	0	−15	0	−23
21	0	−25	24	0	22	23	0	0	0	0	0	0	0	27	0	0	0	−28	0	0	0	5	6	3	−2	0	14	−18
22	−23	0	0	0	−21	0	0	−25	0	24	0	27	0	0	0	0	0	0	−28	0	−5	0	−1	10	−8	0	12	−19
23	22	0	0	0	0	−21	25	0	−24	0	0	0	0	0	27	0	0	0	0	28	−6	1	0	−9	7	0	15	20
24	0	0	−21	25	0	0	0	0	23	−22	27	0	0	0	0	28	0	0	0	0	−3	−10	9	0	4	0	11	16
25	0	21	0	−24	0	0	−23	22	0	0	0	0	27	0	0	0	28	0	0	0	2	8	−7	−4	0	0	13	17
26	0	0	0	0	0	0	0	0	0	0	16	−19	17	−18	20	11	13	−14	−12	15	0	0	0	0	0	0	28	27
27	0	0	0	0	0	0	0	0	0	0	−24	−22	−25	−21	−23	0	0	0	0	0	−14	−12	−15	−11	−13	−28	0	−26
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	25	−21	−22	23	18	19	−20	−16	−17	−27	26	0

Appendix C

The components of the momentum map K.

$$
\begin{aligned}
2K_1 &= -y_0x_1 + y_1x_0 + y_2x_3 - y_3x_2 - y_4x_5 + y_5x_4 + y_6x_7 - y_7x_6,\\
2K_2 &= -y_0x_1 + y_1x_0 + y_2x_3 - y_3x_2 + y_4x_5 - y_5x_4 - y_6x_7 + y_7x_6,\\
2K_3 &= -y_0x_5 + y_1x_4 + y_2x_7 - y_3x_6 - y_4x_1 + y_5x_0 + y_6x_3 - y_7x_2,\\
2K_4 &= -y_0x_4 - y_1x_5 - y_2x_6 - y_3x_7 + y_4x_0 + y_5x_1 + y_6x_2 + y_7x_3,\\
2K_5 &= -y_0x_2 - y_1x_3 + y_2x_0 + y_3x_1 - y_4x_6 - y_5x_7 + y_6x_4 + y_7x_5,\\
2K_6 &= -y_0x_3 + y_1x_2 - y_2x_1 + y_3x_0 - y_4x_7 + y_5x_6 - y_6x_5 + y_7x_4,\\
2K_7 &= -y_0x_2 - y_1x_3 + y_2x_0 + y_3x_1 + y_4x_6 + y_5x_7 - y_6x_4 - y_7x_5,\\
2K_8 &= -y_0x_3 + y_1x_2 - y_2x_1 + y_3x_0 + y_4x_7 - y_5x_6 + y_6x_5 - y_7x_4,\\
2K_9 &= -y_0x_6 - y_1x_7 + y_2x_4 + y_3x_5 - y_4x_2 - y_5x_3 + y_6x_0 + y_7x_1,\\
2K_{10} &= -y_0x_7 + y_1x_6 - y_2x_5 + y_3x_4 - y_4x_3 + y_5x_2 - y_6x_1 + y_7x_0,\\
4K_{11} &= -y_0^2 - y_1^2 - y_2^2 - y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2\\
&\quad -x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2,\\
2K_{12} &= y_0y_7 - y_1y_6 + y_2y_5 - y_3y_4 + x_0x_7 - x_1x_6 + x_2x_5 - x_3x_4,\\
2K_{13} &= -y_0y_4 - y_1y_5 - y_2y_6 - y_3y_7 - x_0x_4 - x_1x_5 - x_2x_6 - x_3x_7,\\
2K_{14} &= y_0y_5 - y_1y_4 - y_2y_7 + y_3y_6 + x_0x_5 - x_1x_4 - x_2x_7 + x_3x_6,\\
2K_{15} &= -y_0y_6 - y_1y_7 + y_2y_4 + y_3y_5 - x_0x_6 - x_1x_7 + x_2x_4 + x_3x_5,\\
2K_{16} &= y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 - y_4x_4 - y_5x_5 - y_6x_6 - y_7x_7,\\
2K_{17} &= y_0x_4 + y_1x_5 + y_2x_6 + y_3x_7 + y_4x_0 + y_5x_1 + y_6x_2 + y_7x_3,\\
2K_{18} &= y_0x_5 - y_1x_4 - y_2x_7 + y_3x_6 - y_4x_1 + y_5x_0 + y_6x_3 - y_7x_2,\\
2K_{19} &= y_0x_7 - y_1x_6 + y_2x_5 - y_3x_4 - y_4x_3 + y_5x_2 - y_6x_1 + y_7x_0,\\
2K_{20} &= y_0x_6 + y_1x_7 - y_2x_4 - y_3x_5 - y_4x_2 - y_5x_3 + y_6x_0 + y_7x_1,\\
2K_{21} &= y_0y_5 - y_1y_4 - y_2y_7 + y_3y_6 - x_0x_5 + x_1x_4 + x_2x_7 - x_3x_6,\\
2K_{22} &= y_0y_7 - y_1y_6 + y_2y_5 - y_3y_4 - x_0x_7 + x_1x_6 - x_2x_5 + x_3x_4,\\
2K_{23} &= -y_0y_6 - y_1y_7 + y_2y_4 + y_3y_5 - x_0x_6 + x_1x_7 - x_2x_4 - x_3x_5,\\
4K_{24} &= -y_0^2 - y_1^2 - y_2^2 - y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2\\
&\quad +x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2,\\
2K_{25} &= -y_0y_4 - y_1y_5 - y_2y_6 - y_3y_7 + x_0x_4 + x_1x_5 + x_2x_6 + x_3x_7,\\
4K_{26} &= -y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 - y_5^2 - y_6^2 - y_7^2\\
&\quad +x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2,\\
2K_{27} &= y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4 + y_5x_5 + y_6x_6 + y_7x_7,\\
4K_{28} &= -y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 - y_5^2 - y_6^2 - y_7^2\\
&\quad -x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2.
\end{aligned}
$$

Appendix D

The proof of Lemma 1. First we define a quadratic form on $\mathbf{H}^4$ by

$$\phi(u, v) = uGv^*, \quad G = \mathrm{diag}(\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i}), \tag{D1}$$

where $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4) \in \mathbf{H}^4$. It is clear that the momentum map associated with the $Sp(1)$ left action is expressed as $\Phi(w) = -\frac{1}{2}\phi(w, w)$ (see (59)). A simple calculation provides the relations

$$\phi(u, v) = -\phi(v, u)^*, \qquad \phi(q_1 u, q_2 v) = q_1 \phi(u, v) q_2^*, \quad q_1, q_2 \in \mathbf{H}. \tag{D2}$$

In order to prove the transitivity of $Sp(2\mathrm{i}, 2\mathrm{i})$ on $\Phi^{-1}(\mu)$, it suffices to show the transitivity on $\Phi^{-1}(-\mathrm{i}/2)$. This is because for an arbitrary non-zero $\mu \in sp(1)$, the left action $w \mapsto qw$ with a quaternion $q \in \mathbf{H}$ satisfying $-q(\mathrm{i}/2)q^* = \mu$ maps diffeomorphically $\Phi^{-1}(-\mathrm{i}/2)$ to $\Phi^{-1}(\mu)$, and because the left action is commutative with the $Sp(2\mathrm{i}, 2\mathrm{i})$ right action.

Let us take an arbitrary point w in $\Phi^{-1}(-\mathrm{i}/2)$. Then one has $\phi(w, w) = \mathrm{i}$. Further, we set

$$p_1 = w, \quad p_2 = (0, 1, 0, 0), \quad p_3 = (0, 0, 1, 0), \quad p_4 = (0, 0, 0, 1). \tag{D3}$$

Using these vectors, we define the following vectors:

$$\begin{aligned} &a_1 = p_1, \quad a_2 = p_2 - \phi(a_1, p_2)^* \mathrm{i} a_1, \quad a_3 = p_3 - \phi(a_1, p_3)^* \mathrm{i} a_1 - \phi(a_2, p_3)^* \mathrm{i} a_2, \\ &a_4 = p_4 - \phi(a_1, p_4)^* \mathrm{i} a_1 - \phi(a_2, p_4)^* \mathrm{i} a_2 - \phi(a_3, p_4)^* \mathrm{i} a_3. \end{aligned} \tag{D4}$$

Then we obtain

$$\phi(a_1, a_1) = \mathrm{i}, \quad \phi(a_1, a_2) = \phi(a_1, a_3) = \phi(a_1, a_4) = 0. \tag{D5}$$

We proceed to define the following vectors:

$$\begin{aligned} &b_1 = a_1, \quad b_2 = h_2 a_2, \quad b_3 = a_3 - \phi(b_2, a_3)^* \mathrm{i} b_2, \\ &b_4 = a_4 - \phi(b_2, a_4)^* \mathrm{i} b_2 - \phi(b_3, a_4)^* \mathrm{i} b_3, \end{aligned} \tag{D6}$$

where $h_2 \in \mathbf{H}$ should be chosen so that $\phi(h_2 a_2, h_2 a_2) = \mathrm{i}$ may hold. In case of $\phi(a_2, a_2) = 0$, we can change the definition of a_2 so that $\phi(a'_2, a'_2) \neq 0$ may hold for a new a'_2, e.g., p_2 is replaced by $\mathrm{j}p_2$ in the definition of a_2. Then one has

$$\begin{aligned} &\phi(b_1, b_1) = \mathrm{i}, \quad \phi(b_1, b_2) = \phi(b_1, b_3) = \phi(b_1, b_4) = 0, \quad \phi(b_2, b_2) = \mathrm{i}, \\ &\phi(b_2, b_3) = \phi(b_2, b_4) = 0. \end{aligned} \tag{D7}$$

Our next step is to define the following vectors:

$$c_1 = b_1, \quad c_2 = b_2, \quad c_3 = h_3 b_3, \quad c_4 = b_4 + \phi(c_3, b_4)^* \mathrm{i} c_3, \tag{D8}$$

where $h_3 \in \mathbf{H}$ should be chosen so that $\phi(h_3 b_3, h_3 b_3) = -\mathrm{i}$ may hold. Then we find that

$$\begin{aligned} &\phi(c_1, c_1) = \mathrm{i}, \quad \phi(c_1, c_2) = \phi(c_1, c_3) = \phi(c_1, c_4) = 0, \quad \phi(c_2, c_2) = \mathrm{i}, \\ &\phi(c_2, c_3) = \phi(c_2, c_4) = 0, \quad \phi(c_3, c_3) = -\mathrm{i}, \quad \phi(c_3, c_4) = 0. \end{aligned} \tag{D9}$$

Finally, we choose $h_4 \in \mathbf{H}$ so that $\phi(h_4 c_4, h_4 c_4) = -\mathrm{i}$ may hold, and set

$$g_1 = c_1, \quad g_2 = c_2, \quad g_3 = c_3, \quad g_4 = h_4 c_4. \tag{D10}$$

In conclusion, we obtain

$$\phi(g_n, g_m) = G_{nm}, \quad n, m = 1, \dots, 4, \tag{D11}$$

where G_{nm} are components of the matrix G. This equation means that the matrix g having the row vectors g_n, $n = 1, \dots, 4$, defined above is in $Sp(2\mathrm{i}, 2\mathrm{i})$.

By definition, we have

$$(1, 0, 0, 0)g = g_1 = w, \tag{D12}$$

which means that $Sp(2\mathrm{i}, 2\mathrm{i})$ acts transitively on $\Phi^{-1}(-i/2)$ and hence on $\Phi^{-1}(\mu)$. This completes the proof.

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